WHAT DISTINGUISHES AN INDEPENDENTLY OBSERVED VECTOR FROM AN ESTIMATED MULTIVARIATE NORMAL POPULATION?

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SUMMARY

Once a researcher has determined that a multivariate observation $\underline{x}_o$ is different from an estimated population, he still has an unanswered question. He wants to know, "How is it different?" A method for answering this question is considered in this report.

Two results are shown and both involve the estimated population parameters $\overline{x}$-the estimated mean, an $\hat{\Sigma}$-the estimated covariance matrix. The first result answers the question "Which linear combinations of the elements of the difference $\underline{x}_o - \overline{x}$ are significant?" The second answers the question "Which elements of the difference $\underline{x}_o - \overline{x}$ are significant?" Both results are derived using S. N. Roy's Union-Intersection Principle; hence, one can set an overall α/significance/type-1 level for the totality of tests.

A brief discussion of an application is also presented.

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GLOSSARY

$\underline{x}_o$ A p-component (p by 1) vector observation, "o", independent of the vectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$

n   The number of independent observations which are independent of the vector $\underline{x}_o$

$\bar{\underline{x}} = \frac{1}{n} \sum_{i=1}^{n} \underline{x}_i$ A p-component (p by 1) estimate of the population mean vector $\underline{\mu}$

$\underline{\mu}$ A p-variate (p by 1) population mean vector

$\underline{\Sigma}$ A p by p population covariance matrix

$\hat{\underline{\Sigma}}$ An unbiased estimate of the population covariance matrix based on m degrees of freedom

$\hat{\underline{\Sigma}}^{-1}$ The inverse of the matrix $\hat{\underline{\Sigma}}$

a The probability that the (null) hypothesis H_0 will be rejected when it is true

H_0 The hypothesis that the vector $\underline{x}_o$ and the set of vectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ are from the same population

H_1 The negation of the hypothesis H_0

$N(\underline{\mu}, \underline{\Sigma})$ The multivariate normal population with parameters $\underline{\mu}$ and $\underline{\Sigma}$

m   The number of degrees of freedom of the matrix estimate $\hat{\underline{\Sigma}}$

t   A "students" t-test statistic

$\underline{c}$ A (1 by p) p-variate vector of constant coefficients

$\underline{c}^t$ The p by 1 transpose of the vector $\underline{c}$

$\underline{\tilde{c}}$ The (1 by p) p-variate vector which maximizes the function $T^2(\underline{c})$

$\underline{e}_i$ A 1 by p vector which has a 1 as the i-th element and zeros elsewhere

$\dfrac{\partial T^2(\underline{c})}{\partial(\underline{c})}$ A p-variate (p by 1) vector of partial derivatives whose i-th component is $\dfrac{\partial T^2(\underline{c})}{\partial(c_i)}$

$\underline{0}$ A p by 1 vector which has zeros for all the elements

s_i^2 An unbiased estimate of the i-th variate from $N(\underline{\mu}, \Sigma)$

D   A matrix conformal with $\underline{x}$

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INTRODUCTION

In a previous report by the author (reference 1), a probability density/distribution function was developed for an observed (p by 1) vector ($\underline{x}_o$) from a multivariate normal distribution with estimated parameters (see A-1).* This result enables a researcher to ask the general question "How unlikely is it that the observation $\underline{x}_o$ arose from $N(\underline{\mu}, \underline{\Sigma})$ where $\underline{\mu}$ is estimated by $\overline{\underline{x}}$ and $\underline{\Sigma}$ is estimated by $\hat{\underline{\Sigma}}$?" Specifically the statistic:

$$T^2 = \frac{n}{n+1} (\underline{x}_o - \overline{\underline{x}})^t \hat{\underline{\Sigma}}^{-1} (\underline{x}_o - \overline{\underline{x}}) \tag{1}$$

which is distributed as $\dfrac{mp}{m - p + 1}$ F with p and m-p+1 degrees of freedom (df), can be compared with tables of the F distribution for probability of occurrence. If the probability of occurrence is less than some specified amount (a), then the hypothesis (H_0) that $\underline{x}_o$ is from the same population that generated $\overline{\underline{x}}$ could be rejected. In other words, the hypothesis (H_1) that the populations which generated $\underline{x}_o$ and $\overline{\underline{x}}$ are not the same would be accepted.

The rejection of the hypothesis that $\underline{x}_o$ and $\overline{\underline{x}}$ are from the same population (H_0) doesn't show which of the variates of $\underline{x}_o$ were significantly different. Although individual tests of significance could be constructed (e.g., t-tests), there is no control of the overall a level for the set of comparisons. The reasons for this are twofold; there are p such tests, and the variates are generally correlated. Hence an approach is needed for testing the individual variates of $\underline{x}_o$ while controlling the overall a-level. The purpose of the present development is to delineate such a procedure.

*The results A-1 and A-2 are working theorems which are given in the appendix.

APPROACH

The approach employed here uses S.N. Roy's Union-Intersection Principle (reference 2, and reference 3). This principle allows one to fix the overall significance level for the totality of tests of linear compounds of the difference $\underline{x}_o - \overline{\underline{x}}$. Operationally, this is accomplished by employing the same (a-level) criterion required for the test of the most unlikely linear compound, $\underline{\tilde{c}}(\underline{x}_o - \overline{\underline{x}})$, to all particular tests of linear compounds. Since the individual variates $x_{oi} - \overline{x}_i$ (i=1, . . . , p) can be tested by the linear compounds $\underline{e}_i(\underline{x}_o - \overline{\underline{x}})$ (i=1, . . . , p), this procedure will yield a solution to the problem posed in the Introduction.*

Let us define the statistic $T^2(\underline{c})$ as follows:

$$T^2(\underline{c}) = \frac{n}{n+1}\left[\underline{c}(\underline{x}_o - \overline{\underline{x}})\right]^t\left[\underline{c}\,\hat{\Sigma}\,\underline{c}^t\right]^{-1}\left[\underline{c}(\underline{x}_o - \overline{\underline{x}})\right]$$ (2)

where $\underline{c}$ is a fixed 1 by p vector. This equation is a special case of equation (1) with the p-variate terms $\underline{x}_o, \overline{\underline{x}}$, and $\hat{\Sigma}$ replaced by the corresponding univariate terms $\underline{c}\underline{x}_o$, $\underline{c}\underline{x}$, and $\underline{c}\,\hat{\Sigma}\,\underline{c}^t$. These univariate terms are those appropriate for linear compounds of the respective variates (see A-2). Under the hypothesis (H_0) that $\underline{x}_o$ is from the same population that generated $\overline{\underline{x}}$, equation (2) is distributed as F with 1 and m degrees of freedom. Hence, the most unlikely $T^2(\underline{c})$ value would occur for that $\underline{c}$ which maximizes (2).

DERIVATIONS

In the following, a theorem will be stated which contains both the conditions for maximizing $T^2(\underline{c})$, and a procedure for testing the significance of the totality of linear compounds with a fixed overall significance level. Consider the following:

Theorem 1.0. If $T^2(\underline{c})$ is defined as in equation (2), then its maximum value is

$$T^2(\underline{\tilde{c}}) = \frac{n}{n+1}(\underline{x}_o - \overline{\underline{x}})^t\hat{\Sigma}^{-1}(\underline{x}_o - \overline{\underline{x}})$$ (3)

which is distributed as $\dfrac{mp}{m-p+1}$ F with p and m-p+1 degrees of freedom when H_0 is true. This value of $T^2(\underline{c})$ is obtained for

$$\underline{\tilde{c}} = (\underline{x}_o - \overline{\underline{x}})^t\hat{\Sigma}^{-1}$$ (4)

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*$\underline{e}_i$ (i=1, . . ., p) is a 1 by p vector with a 1 in the i-th entry and zeros elsewhere.

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Further, the totality of linear compounds of the form $\underline{c}(\underline{x}_o - \overline{\underline{x}})$ could be tested for significance at the overall a-level by the test:

$$T^2(\underline{c}) \begin{array}{c} H_1 \\ \gtrless \\ H_0 \end{array} \frac{mp}{m-p+1} F_{a:p,m-p+1} \tag{5}$$

Here H_1 is accepted if $T^2(\underline{c})$ is greater than the product of $(mp)/(m-p+1)$ and the a-level F for p and m-p+1 degrees of freedom and H_0 is accepted otherwise.

Proof. Let is first rewrite equation (2) as follows:

$$T^2(\underline{c}) = \frac{n}{n+1} \frac{\underline{c}(\underline{x}_o - \overline{\underline{x}})(\underline{x}_o - \overline{\underline{x}})^t \underline{c}^t}{\underline{c}\hat{\Sigma}\underline{c}^t} \tag{6}$$

To obtain the condition for maximizing $T^2(\underline{c})$, let us take its partial derivative with respect to $\underline{c}$ and set the resulting system equal to $\underline{0}$. From the rules of differentation, it can be seen that

$$\frac{\partial T^2(\underline{c})}{\partial \underline{c}} = \frac{n}{n+1}(\underline{c}\hat{\Sigma}\underline{c}^t)^{-2}\Big[\underline{c}\hat{\Sigma}\underline{c}^t\,[2(\underline{x}_o - \overline{\underline{x}})(\underline{x}_o - \overline{\underline{x}})^t\,\underline{c}^t]$$

$$- \underline{c}(\underline{x}_o - \overline{\underline{x}})(\underline{x}_o - \overline{\underline{x}})^t\underline{c}^t\,[2\hat{\Sigma}\underline{c}^t]\Big] \tag{7}$$

Setting this system equal to $\underline{0}$ and solving, one can obtain the condition:

$$[\hat{\Sigma}^{-1}(\underline{x}_o - \overline{\underline{x}})(\underline{x}_o - \overline{\underline{x}})^t - \lambda I_p]\underline{c}^t = \underline{0} \tag{8}$$

where

$$\lambda = \frac{\underline{c}(\underline{x}_o - \overline{\underline{x}})(\underline{x}_o - \overline{\underline{x}})^t\underline{c}^t}{\underline{c}\hat{\Sigma}\underline{c}^t} \tag{9}$$

It is apparent upon comparison of (9) with (6) that the maximum $T^2(\underline{c})$ would be obtained for the transposed eigenvector which corresponds to the largest eigenvalue of the matrix:

$$\frac{n}{n+1}\hat{\Sigma}^{-1}(\underline{x}_o - \overline{\underline{x}})(\underline{x}_o - \overline{\underline{x}})^t \tag{10}$$

There will be one nonzero eigenvalue of (10) since the rank of $(\bar{x}_o - \bar{x})(x_o - \bar{x})^t$ is one and the product of it and any conformal nonsingular matrix (e.g., $\hat{\Sigma}^{-1}$) would have the same rank.[*] Since the trace (tr) of (10) equals the sum of its eigenvalues (of which there is exactly one, $T^2(\tilde{c})$, that is nonzero), it follows that

$$T^2(\tilde{c}) = \text{tr}\left[\frac{n}{n+1} \hat{\Sigma}^{-1}(x_o - \bar{x})(x_o - \bar{x})^t\right] \tag{11}$$

This can, by the commutative laws for traces, be rewritten

$$T^2(\tilde{c}) = \left(\frac{n}{n+1}\right) \text{tr}\left[(x_o - \bar{x})^t \hat{\Sigma}^{-1}(x_o - \bar{x})\right] \tag{12}$$

$$= \left(\frac{n}{n+1}\right)(x_o - \bar{x})^t \hat{\Sigma}^{-1}(x_o - \bar{x}) \tag{13}$$

which is the first result (3) of Theorem 1.0.[*] The distribution of (3) or (13) is given by A-1; hence, the first result of the theorem is completed.

The second result of this theorem $(\tilde{c})$ follows upon substitution of the value of $\tilde{c}$ given in equation (4) for c in equation (6). This yields a $T^2(c)$ value of

$$\left(\frac{n}{n+1}\right)\frac{\left[(x_o - \bar{x})^t \hat{\Sigma}^{-1}\right](x_o - \bar{x})(x_o - \bar{x})^t \left[(x_o - \bar{x})^t \hat{\Sigma}^{-1}\right]^t}{(x_o - \bar{x})^t \hat{\Sigma}^{-1}(x_o - \bar{x})} \tag{14}$$

or equivalently

$$\left(\frac{n}{n+1}\right)(x_o - \bar{x})^t \hat{\Sigma}^{-1}(x_o - \bar{x}) \tag{15}$$

Because this corresponds to the maximum value of $T^2(c)$, equation (5) gives the desired vector.

The last portion of the theorem can be seen when one recalls the nature of the Union-Intersection principle. Specifically, by employing the significance criterion (a - level) of $T^2(\tilde{c})$ for each test of the type $T^2(c)$, one is assured that the totality of such tests has a joint significance level of a. Since $\frac{mp}{m - p+1} F_{a:p,m-p+1}$ is this criteria, as indicated by the first part of the theorem and A-1, the general test (5) follows directly.

[*]See reference 4 for discussion of the rank of the product of two matrices.

[**]Reference 4 also contains a discussion of various results concerning the traces of matrices.

Tests for the individual components can be derived from specialization of equation (5). Since testing a specific component for significance (i-th) is equivalent to testing $T^2(\underline{e}_i)$ for significance, one could substitute $\underline{e}_i$ for $\underline{c}$ in equation (5) and obtain a specialized result. This will be considered in the remainder of this section.

Substitution of $\underline{e}_i$ for $\underline{c}$ in equation (5) yields

$$T^2(\underline{e}_i) \underset{H_0}{\overset{H_1}{\underset{\leq}{\geq}}} \frac{mp}{m-p+1} F_{a:p,m-p+1} \tag{16}$$

which by equation (2) is equivalent to

$$\left(\frac{n}{n+1}\right)\left[\underline{e}_i(\underline{x}_o - \overline{\underline{x}})\right]^t \left[\underline{e}_i \, \hat{\Sigma} \, \underline{e}_i^t\right]^{-1} \left[\underline{e}_i(\underline{x}_o - \overline{\underline{x}})\right] \underset{H_0}{\overset{H_1}{\underset{\leq}{\geq}}} \frac{mp}{m-p+1} F_{a:p,m-p+1} \tag{17}$$

Multiplying out each of the bracketed terms,

$$\left(\frac{n}{n+1}\right)\left[x_{oi} - \overline{x}_i\right]\left[\hat{\Sigma}_{ii}\right]^{-1}\left[x_{oi} - \overline{x}_i\right] \underset{H_0}{\overset{H_1}{\underset{\leq}{\geq}}} \frac{mp}{m-p+1} F_{a:p,m-p+1} \tag{18}$$

where x_{oi} is the i-th entry of $\underline{x}_o$, $\overline{x}_i$ is the i-th entry of $\overline{\underline{x}}$, and $\hat{\Sigma}_{ii}$ is the i, i-th entry of $\hat{\Sigma}$. Since in general the diagonal entries of $\hat{\Sigma}$ are variance estimates, $[\hat{\Sigma}_{ii}]^{-1}$ can be written as $\frac{1}{s_i^2}$ where s_i^2 is the estimated variance of the i-th variate. Thus equation (18) can be written:

$$\left(\frac{n}{n+1}\right)\frac{(x_{oi} - \overline{x}_i)^2}{s_i^2} \underset{H_0}{\overset{H_1}{\underset{\leq}{\geq}}} \frac{mp}{m-p+1} F_{a:p,m-p+1} \tag{19}$$

which is a considerable simplification of the test equation of the theorem. This result is summarized in the following corollary:

Corollary 1.1 Let x_{oi} be the i-th entry of $\underline{x}_o$, $\overline{x}_i$ be the i-th entry of $\overline{\underline{x}}$, and s_i^2 be the i, i-th entry of $\hat{\Sigma}$. Then the set of p-variates of $\underline{x}_o$ can be individually tested for difference from $\underline{\mu}$ by

$$T^2(\underline{e}_i) = \left(\frac{n}{n+1}\right)\frac{(x_{oi} - \overline{x}_i)^2}{s_i^2} \underset{H_0}{\overset{H_1}{\underset{\leq}{\geq}}} \frac{mp}{m-p+1} F_{a:p,m-p+1} \tag{20}$$

with assurance that the overall significance level for the entire set of p tests is less than a.

DISCUSSION

In the above, two results were derived which answered the question "How does the observation $\underline{x}_o$ differ from the population which generated $\overline{\underline{x}}$?" The first of these, Theorem 1.0, allows one to ask if any particular linear combination of variates ($\underline{c}\underline{x}_o$) distinguishes $\underline{x}_o$ from the population which generated $\overline{\underline{x}}$. The second result, Corollary 1.1, allows one to ask if any particular variate distinguishes $\underline{x}_o$ from the $\overline{\underline{x}}$ generating population. Both of these results have obvious practical application. Let us briefly consider one.

In a multiple criteria experiment, an unforseen (random) event occurs which makes suspect a single observation $\underline{x}_o$. The first question which faces the research is, "Is $\underline{x}_o$ from the same population as the set of other observations ($\underline{x}_1$, $\underline{x}_2$, . . . , $\underline{x}_n$) from the same condition?" This question can be answered by employing the statistic:

$$T^2 = \frac{n}{n+1} (\underline{x}_o - \overline{\underline{x}})^t \ \hat{\Sigma}^{-1} (\underline{x}_o - \overline{\underline{x}}) \tag{21}$$

where $\overline{\underline{x}} = \frac{1}{n} \sum_{i=1}^{n} \underline{x}_i$, $\hat{\Sigma} = \frac{1}{n-1} \sum_{i=1}^{n} (\underline{x}_i - \overline{\underline{x}})(\underline{x}_i - \overline{\underline{x}})^t$, and T^2 is distributed as $\frac{(n-1)p}{n-p}$ F with p and n–p degrees of freedom.* Given that this statistic (21) is significant, the next question is, "Which variates of $\underline{x}_o$ differ from the population which generated $\overline{\underline{x}}$ and $\hat{\Sigma}$?" This could be answered by applying Corollary 1.1 with m equaling n–1. Guided by the costs of observations and their number, the results of these tests would be useful for decisions regarding the inclusion of $\underline{x}_o$ as part of the data of the experiment.

The above does not exhaust the set of possible applications. Hopefully this application will suggest others to the reader.

REFERENCES

1. Naval Missile Center. Probability Density When an Independent Observation is From an Estimated Multivariate Normal Population, by A. C. Bittner Jr. Point Mugu, Calif., 15 Jul 1971. (TP-71-33) UNCLASSIFIED

2. Morrison, D. F. Multivariate Statistical Methods, San Francisco, McGraw-Hill, 1967.

3. Roy, S. N. Some Aspects of Multivariate Analysis, New York, Wiley, 1957.

4. Marcus, M. and Minc, H. Introduction to Linear Algebra, New York, Macmillian, 1965.

5. Anderson, T. W. An Introduction to Multivariate Statistical Analysis, New York, Wiley, 1958.

*The statistic shown in (21) is a variation of A-1. A proof for its distribution was shown in reference 1.

APPENDIX

TWO WORKING THEOREMS

The first theorem (A-1) was shown in reference 1 by the author. The second theorem (A-2) was shown by Anderson (reference 5) in 1958.

A-1

If $\underline{x}_o$ is an observed p-variate vector from $N(\underline{\mu}, \underline{\Sigma})$,

$$\bar{\underline{x}} = \frac{1}{n} \sum_{i=1}^{n} \underline{x}_i$$

is a mean vector also from $N(\mu, \underline{\Sigma})$ based on n independent observations, and $m\hat{\underline{\Sigma}}$ is the sum of the matrix products of m independent $N(\underline{0}, \underline{\Sigma})$ p-variate vectors $(\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_m)$; i.e.,

$$m\hat{\underline{\Sigma}} = \sum_{i=1}^{m} \underline{Z}_i \underline{Z}_i^t$$

then

$$T^2 = \frac{n}{n+1} (\underline{x}_o - \bar{\underline{x}})^t \, \hat{\underline{\Sigma}}^{-1} (\underline{x}_o - \bar{\underline{x}})$$

is distributed as:

$$\frac{mp}{m - p + 1} \, F$$

where F has p and m − p + 1 degrees of freedom.

If x is distributed according to $N(\mu, \Sigma)$, then $Z = D\underline{x}$ is distributed according to $N(D\mu, D\Sigma D^t)$.

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13. ABSTRACT

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Two results are shown and both involve the estimated population parameters $\bar{\underline{x}}$-the estimated mean, and $\hat{\underline{\Sigma}}$-the estimated covariance matrix. The first result answers the question "Which linear combinations of the elements of the difference $\underline{x}_o - \bar{\underline{x}}$ are significant?" The second answers the question "Which elements of the difference $\underline{x}_o - \bar{\underline{x}}$ are significant?" Both results are derived using S. N. Roy's Union-Intersection principle; hence, one can set an overall α/significance/type-1 level for the totality of tests.

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